Boundary conditions and consistency of effective theories

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Effective theories are non-local at the scale of the eliminated heavy particles modes. The gradient expansion which represents such non-locality must be truncated to have treatable models. This step leads to the proliferation of the degrees of freedom which renders the identification of the states of the effective theory nontrivial. Furthermore it generates non-definite metric in the Fock space which in turn endangers the unitarity of the effective theory. It is shown that imposing a generalized KMS boundary conditions for the new degrees of freedom leads to reflection positivity for a wide class of Euclidean effective theories, thereby these lead to acceptable theories when extended to real time.

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I. INTRODUCTION

The observed richness of the scale dependence of fundamental physical laws renders the project of constructing the ultimate Theory of Everything unpractical. Instead, effective theories are put forward, models with limited range of applicability, as a less ambitious but more realistic alternative. Such a model is derived from an underlying, more microscopic theory by eliminating the degrees of freedom which belong to short distances not resolved by the effective model.

The elimination of a propagating particle mode from a local theory induces long range correlations in the remaining effective dynamics. It is easy to verify in the framework of imaginary time, Euclidean quantum field theories that these non-local features arising from the elimination of massive particle modes can be retained as higher order derivative terms in the effective action. By restricting our interest to sufficiently low energy the gradient expansion can be truncated. We consider in this work two questions raised by this step, the specification of the states and the consistency of the effective theory.

Let us consider a single degree of freedom which is governed by a Lagrangian containing time derivatives up to n_d-th order. Its classical phase space is $2n_d$ dimensional. Assuming that ordinary, covariant particles are defined by the usual, velocity dependent Lagrangians we may match this space by the phase space of n_d ordinary particles. One arrives at similar conclusions in quantum physics when time derivatives up to order n_d are added to the Schrördinger equation and one introduces n_d component wavefunctions, consisting of the first $n_d - 1$ time derivatives of the original wavefunction. One may gain a new insight into the appearance of antiparticles in relativistic quantum mechanics in this manner. The issue of identifying the states of an effective theory with higher order derivatives is therefore nontrivial. This complication is actually natural because in order to arrive at a well defined effective theory we have to specify the initial conditions for the heavy particles which are eliminated. The specification of the initial or final states amounts to the definition of boundary conditions in time within the path integral formalism. Notice that this problem is bound entirely to the truncation of the effective dynamics. In fact, when the heavy particles with well defined initial conditions are eliminated exactly then the resulting integro-differential equations yield unique solutions when the usual, $n_d = 1$ initial conditions are imposed on the light particles. But we have to keep in mind that no exact equations of motion are known in physics, we should allow the presence of higher order time derivatives in the Lagrangian with some small coefficients in all theories.

The other question related to the truncation is that even if the underlying theory is consistent, follows a unitary time evolution, the effective model, defined by a truncated gradient expansion may show inconsistencies and provide non-unitary time evolution for the light particles. Consider for instance a solid state at low enough temperature, described in terms of electrons and phonons. The periodic ion core potential induces a band structure which can be modelled by a Lagrangian for the electrons without potential but containing higher order derivatives in the kinetic energy. Such a theory becomes unusable at sufficiently high energy where lattice defects may occur. The true physics requires new degrees of freedom which should appear as a loss of unitarity of the simple effective model based on electrons and phonons. We argue below that these two problems are related, an appropriately chosen subspace of the states of the effective theory leads to consistent dynamics in the natural manner.

To see our problems in a simpler setting consider a scalar theory defined by the action

$$S[\phi] = \int dx \left[\phi \left(\sum_{n=1}^{n_d} c_n \Box^n \right) \phi(x) - V(\phi(x)) \right] \tag{1}$$

with derivatives up to order n_d. We assume that the model obeys time reversal invariance, the coefficients c_n and the potential $V(\phi)$ are real, furthermore $(-1)^{n_d} c_{n_d} > 0$. For the sake of simplicity we take n_d an odd integer. The generator functional for the Green functions can be written as

$$\int D[\phi] e^{iS[\phi]+i\int dx j(x)\phi(x)} = e^{-i\int dx V(\frac{\delta}{i\delta j(x)})} e^{-\frac{i}{2}\int dx dy j(x) D(x-y) j(y)} \tag{2}$$

where the free propagator is

$$D(p) = \left(\sum_{n=1}^{n_d} (-1)^n c_n (p^2)^n \right)^{-1} \tag{3}$$

in the momentum-space, $i\epsilon$ prescription being suppressed. Its partial fraction decomposition,

$$D(p) = \sum_{j=1}^{n_d} \frac{Z_j}{p^2 - m_j^2} \tag{4}$$

clearly involves negative contributions, there is at least one Z factor which is negative. The free generator functional, the second exponential factor in Eq. (2), suggests the presence of states with negative norm and may involve complex energy states $m_j^2 < 0$, when the decomposition (4) is used [1]. Though the effective model which retains the dynamics of the eliminated particle modes exactly remains consistent the truncation of the gradient expansion may introduce instability and loss of unitarity. In fact, some particles introduced by the decomposition (4) may have complex energy, $m_j^2 < 0$, and it is not clear if the time evolution remains unitary within the subspace of physical, positive norm states. These questions raise concern about the feasibility of the effective theory strategy. We shall see that the negative norm states provide a common framework to address both issues mentioned above, namely the specification of states and the consistency.

The higher order derivative terms have already been considered in quantum field theory as regulators [2–7] and they are known to generate states with negative norm [8]. An extension of the Higgs sector of the Standard Model involving higher order derivative terms in the kinetic energy has been proposed [9], too. Negative norm states occur in the covariant quantization of gauge theories as well [10, 11] but they can safely be excluded from the asymptotic states by means of gauge symmetry without upsetting the unitary time evolution in the physical, positive norm sector. Without such a powerful symmetry argument the fate of the instability and unitarity becomes a more difficult problem for the effective models to resolve.

It was conjectured that instability and loss of unitarity, generated by ghost particles with complex energy and negative norm can be excluded from the asymptotic states of finite energy because the mass of a ghost particle diverges with the cut-off Λ [12]. The role of the cut-off is taken over the heavy particle energy scale in the context of the effective models. This argument eliminates safely the states with negative norm but leaves a delicate point to settle, the case of zero norm states. These states which arise in a natural manner in a linear space with indefinite metric may develop exponential growth in time in their amplitude and spoil the stability of the model. The proposal, put forward in Refs. [12] is to impose boundary conditions which eliminate the unstable modes. Similar treatment has been pursued in excluding the runaway solution of the radiation reaction problem in classical electrodynamics [13]. Such boundary conditions imposed in the future appear rather ad hoc and it generates acausal effects. To make things worse, the absence of the exponentially growing amplitudes eliminated by such boundary conditions leads to non-unitary time evolution. The perturbative procedure to restore unitarity by the appropriate modification of the imaginary part of the Feynman graphs runs into difficulties [14].

It will be shown in a non-perturbative manner, by means of lattice regularization in time that the method of classical field theory to treat the higher order time derivatives can be carried over to the quantum case. Furthermore, it is found that the coordinates corresponding to even or odd order time derivatives are represented by self- or skew-adjoint operators, both satisfying the standard canonical commutation relations. An important implication of this correspondence is that the positive definite (indefinite) space belongs to self(skew)-adjoint operators and the trajectories in the path integral are periodic (antiperiodic) in time when expectation values are calculated. This structure which is reminiscent of the KMS construction is used to establish consistency in a non-perturbative manner. The key is the demonstration of reflection positivity for the truncated theories in imaginary time. This argument will be demonstrated in the case of an effective Yang-Mills-Higgs models.

Section II is a brief summary of the salient features of quantum mechanics on linear space with indefinite norm. A generic effective theory, the Yang-Mills-Higgs model with higher order derivatives is introduced in Section III. The model is considered in lattice regularization in Euclidean space-time and reflection positivity is demonstrated in the Fock space span by local operators with positive time reversal parity. This property is sufficient to assure the Wightman axioms for the real-time theory [15]. Finally, the summary of our results is given in Section IV.

II. QUANTUM MECHANICS

Quantum mechanics constructed on linear spaces with indefinite norm [1, 12, 14] has a number of unusual features. These are summarized below by paying special attention to the relation of the signature of the norm with the definition of the adjoint of an operator, the closing relations, the canonical commutation relations and the path integral expressions. The obvious motivation of reviewing these properties is to develop simple means to recognize the restrictions of the linear space and operators which lead to a positive definite linear space and physically interpretable structure. The states with negative or vanishing norm appear at intermediate time only, the physical asymptotic states must have positive norm.

The linear space H with positive definite metric is defined by means of the scalar product $\langle u|v\rangle$ satisfying the requirements (i) $\langle u|v\rangle = \langle v|u\rangle^*$, (ii) $\langle u|(a|v\rangle + b|w\rangle) = a\langle u|v\rangle + b\langle u|w\rangle$, (iii) $\langle u|u\rangle \geq 0$, and (iv) $\langle u|u\rangle = 0$ for $|u\rangle = 0$ only. We shall use decomposable spaces with non-definite metric where properties (iii) and (iv) are replaced by the conditions (iii') $H = H_+ + H_-$ where $H_\pm = \{|u\rangle | \langle u|u\rangle \gtrless 0\}$ with $\langle H_+|H_-\rangle = 0$, and (iv') each vector $|u\rangle$ can be written as $|u\rangle = |u_+\rangle + |u_-\rangle$, $\langle u_\pm|x_\pm\rangle \gtrless 0$ in a unique manner [1]. Note that decomposability, (iv'), makes the metric non-degenerate by excluding zero norm states orthogonal to the rest of the space. We assume furthermore that our linear space can be upgraded to a Hilbert space and can be made complete with respect to the scalar product $\langle u|v\rangle' = \langle u_+|v_+\rangle - \langle u_-|v_-\rangle$.

The classification of the operators is a more involved question now because it is based on two non-trivial quadratic forms, the one given by the operators and another one, the scalar product. Let us assume that our linear space is separable and use a basis $\{|n\rangle\}$ where the non-definite metric $\eta_{mn} = \langle m|n\rangle$ is a non-degenerate Hermitian matrix, $\eta^\dagger = \eta$, which can be brought into the normalized diagonal form $\eta_{mn} = \pm\delta_{m,n}$ by the choice of an appropriate basis. The matrix elements A_{jk} of an operator A are defined in this basis by the equation

$$\langle m|A|n\rangle = \sum_k \eta_{mk} A_{kn}. \tag{5}$$

The non-trivial metric makes the adjoint and the Hermitian adjoint of an operator different. The adjoint $\bar{A}$ of an operator A is defined by $\langle u|\bar{A}|v\rangle = \langle v|A|u\rangle^*$ what gives $\bar{A} = \eta^{-1}A^\dagger\eta$. The self- or skew-adjoint operators satisfy the condition $\bar{A} = \sigma_A A$ with the sign $\sigma_A = +1$ and -1, respectively. Two eigenvectors, $A|\lambda\rangle = \lambda|\lambda\rangle$, $A|\rho\rangle = \rho|\rho\rangle$ give

$$(\lambda - \sigma_A\rho^*)\langle\rho|\lambda\rangle = 0. \tag{6}$$

Thus the spectrum is real or imaginary for self- or skew-adjoint operators, respectively in the subspace of orthogonal eigenvectors with non-vanishing norm. Furthermore, two eigenstates $|\lambda\rangle$ and $|\rho\rangle$ can have non-vanishing overlap only if their eigenvalues are related, $\lambda = \sigma_A\rho^*$, allowing real spectrum for skew-adjoint operators with non-orthogonal eigenvectors.

Let us now consider a free particle whose dynamics is based on the canonical pair of operators $\hat{q}_\sigma$ and $\hat{p}_\sigma$ satisfying $[\hat{q}_\sigma, \hat{p}_\sigma] = i$. These operators are either self- or skew-adjoint and by insisting on real spectrum we have to give up the orthogonality of the eigenstates in the skew-adjoint case and have to use $\eta(q, q') = \delta(q - \sigma q')$ according to Eq. (6). We shall need the closing relation in coordinate basis

$$\mathbb{1} = \int dq|\sigma q\rangle\langle q|. \tag{7}$$

The equations $e^{i\hat{p}q'}\hat{q}e^{-i\hat{p}q'} = \hat{q} + q'$ and $\langle q|p\rangle = e^{ipq}/\sqrt{2\pi}$ yield

$$\eta(p, p') = \langle p|p'\rangle = \int \frac{dq}{2\pi} e^{-iq(p-\sigma p')} = \delta(p - \sigma p') \tag{8}$$

and the closing relation

$$\mathbb{1} = \int dp|\sigma p\rangle\langle p| \tag{9}$$

in momentum space. The self-adjoint operators obviously realize linear space with definite norm. The Pauli matrix σ_x has eigenvalues ± 1, showing that the skew-adjoint case leads to indefinite norm.

Anticipating applications in quantum field theory let us consider a harmonic oscillator of Hamiltonian

$$\hat{H}_\sigma = \frac{\sigma}{2}(\hat{p}_\sigma^2 + \hat{q}_\sigma^2) = \sigma \bar{a}_\sigma a_\sigma, \tag{10}$$

given in terms of the operator $a_\sigma = (\hat{q}_\sigma + i\hat{p}_\sigma)/\sqrt{2}$ satisfying the commutation relation $[a_\sigma, \bar{a}_\sigma] = \sigma$. One can easily construct the Hilbert space where these operators act in an irreducible manner. It is enough to treat the case $\sigma = +1$, the other linear space will be given by the exchange $a \leftrightarrow \bar{a}$. We take the operators $b = a_+$, $\bar{b} = \bar{a}_+$ and start with the usual assumption, the existence of an eigenstate of the self-adjoint operator $\bar{b}b$, $\bar{b}b|\lambda\rangle = \lambda|\lambda\rangle$ and consider the double infinite series of states $\cdots, b^2|\lambda\rangle, b|\lambda\rangle, |\lambda\rangle, \bar{b}|\lambda\rangle, \bar{b}^2|\lambda\rangle, \cdots$ corresponding to the eigenvalues $\cdots, \lambda-2, \lambda-1, \lambda, \lambda+1, \lambda+2, \cdots$ of $\bar{b}b$. If λ is non-integer then this series is infinite on both ends and the Hamiltonian is unbounded. Bounded Hamiltonian requires that the series stops, either to the left or to the right. The equations $\langle\lambda|\bar{b}b|\lambda\rangle = \lambda\langle\lambda|\lambda\rangle$ and $\langle\lambda|b\bar{b}|\lambda\rangle = (\lambda+1)\langle\lambda|\lambda\rangle$ require λ to be integer and the stopping at the left or the right end corresponds to the series $\lambda \geq 0$ or $\lambda \leq -1$, respectively. In the former case we find definite norm, $\mathrm{sign}(\langle\lambda+1|\lambda+1\rangle) = \mathrm{sign}(\langle\lambda|\lambda\rangle)$ and the latter implies norm with both signs, $\mathrm{sign}(\langle\lambda-1|\lambda-1\rangle) = -\mathrm{sign}(\langle\lambda|\lambda\rangle)$. Therefore the canonical operator algebra realized with self- or skew-adjoint operators corresponds to definite or indefinite norm, respectively and the Hamiltonian (10) possesses a stable ground state.

We may find yet another characterization of the signature of the metric in the linear space of states. The time reversal Θ is an anti-unitary transformation, acting in the Schrödinger representation as $\Theta : A \to \bar{A}$ on the operators and one may consider operators with well defined time reversal parity $\Theta A = \tau_A A$, with $\tau_A = \pm 1$. In the Heisenberg representation the transformation properties of the Heisenberg equation of motion under time reversal requires the slight extension $\Theta : A(t) \to \bar{A}(-t)$, in particular the time derivative contributes to the time inversion parity by -1, $\tau_{\partial_0 A} = -\tau_A$. The Heisenberg commutation relations require that the canonically conjugated pairs share the same time reversal parity. The time parity of the canonical operators τ_q determines the signature of the linear space because $\tau_A = \sigma_A$.

Finally, we consider the path integral representation of the time evolution amplitude for a system with a self-adjoint $\hat{q}$ and a skew-adjoint $\hat{q}'$ coordinate,

$$\langle q_f, -q'_f|e^{-itH}|q_i, q'_i\rangle = \int D[p]D[p']D[q]D[q']e^{i\int dt[p\dot{q}+p'\dot{q}'-H(q,q',p,p')]}, \tag{11}$$

where $H(q,q',p,p') = \langle q,q'|\hat{H}|p,p'\rangle/\langle q,q'|p,p'\rangle$ is a complex function for self-adjoint Hamilton operator $\hat{H}$, $H^*(q,q',p,p') = \langle p,-p'|H|q,q'\rangle/\langle p,-p'|q,q'\rangle = H(q,-q',p,-p')$. We regain real $H(q,q',p,p')$ and the phase space path integral can be rendered well defined by the usual $i\epsilon$ prescription for time reversal invariant dynamics. The integration over the momentum trajectories can easily be carried out for the Hamiltonian of the type $\hat{H} = (\hat{p}^2 - \hat{p}'^2)/2 + U(\hat{q},\hat{q}')$. Note the unusual sign in the kinetic energy of the degree of freedom represented by with the skew-adjoint operators, as a result of the coefficient σ in the Hamiltonian (10). As a result, the path integral in the coordinate space is

$$\langle q_f, -q'_f|e^{-itH}|q_i, q'_i\rangle = \int D[q]D[q']e^{i\int dt[\frac{1}{2}\dot{q}^2 - \frac{1}{2}\dot{q}'^2 - U(q,q')]} \tag{12}$$

and it displays an unusual sign in the kinetic energy of this degree of freedom.

III. REFLECTION POSITIVITY

We address now the conditions of arriving at consistent dynamics in a model with higher order time derivatives. We do it in the context of the Yang-Mills-Higgs model for imaginary time with a scalar matter field whose kinetic energy contains higher order derivatives. The action is written as

$$S[\phi, \phi^\dagger, A] = \int d^dx \left[K(D) - \phi^\dagger L(D^2)D^2\phi + V(\phi^\dagger\phi)\right], \tag{13}$$

where the gauge field is given as $A_\mu = A_\mu^a \tau^a$, τ^a denoting the generators of the gauge group, $D_\mu = \partial_\mu - iA_\mu$ stands for the covariant derivative and Λ is the UV cut-off. The functions K and L are bounded from below and are chosen to be a polynomial of the covariant derivative of order at most n_d and $n_d - 2$, respectively. For instance, the perturbatively renormalizable theory is realized by the choice $K(D) = -\mathrm{tr}([D_\mu, D_\nu])^2/2g^2$ and $L = 1$.

The canonical treatment of a classical dynamical systems with higher order time derivatives has been worked out for a long time [16], it is based on the introduction of a new coordinate, together with its canonical pair for each higher order derivative except for the last one, $A_{j\mu}(x) = D_0^j A_\mu(x)$ and $\phi_j(x) = \partial_0^j \phi(x)$ for $j = 0, \ldots, n_d - 1$ in our case. It will be shown below that this apprach is appropriate for quantum fields as well by explicitly constructing the path integral for the fields $A_{j\mu}(x)$ and $\phi_j(x)$.

The higher derivative terms in the action (13) indicate the presence of states with negative norm in the Fock space. The time reversal parity of the coordinates $A_{j\mu}(x)$ and $\phi_j(x)$ is $\tau = (-1)^{j+\delta_{\mu,0}}$ and $(-1)^j$, respectively. The states $\mathcal{F}|0\rangle$ where $\mathcal{F}$ is a local gauge invariant functional of the fields of even time reversal parity should span a Fock space of definite norm. This argument is naturally formal because it is just the difficulty with the definiteness of the norm which prevents us from establishing the quantum counterpart of this classical approach in a reliable manner. Nevertheless its conclusion remains valid as will be shown below. In what follows we use the action (13) directly in the path integral formalism where it is usually derived instead of extending the quantization procedure for higher order derivatives.

The strategy to give meaning to the theory (13) is to find a set of fields whose Green functions satisfy Wightman's axioms when continued to real time. The set of conditions which is necessary and sufficient for this to happen has been found [15] and its piece which requires explicit justification is reflection positivity. It is usually tested in lattice regularization where it appears as the positivity of the transfer matrix in imaginary time. Therefore we consider the model for imaginary time where lattice regularization is imposed. What is crucial is the introduction of a finite step size in time, the spatial directions will be discretized only because there is no other way to regulate gauge theories in a non-perturbative manner. The lattice fields are $\phi(n) = a\phi(x)$, $\phi^\dagger(n) = a\phi^\dagger(x)$, $U_\mu(n) = U^\dagger_{-\mu}(n+\hat\mu) = e^{igaA_\mu(n)}$, with a as lattice spacing and $\hat\mu$ denotes the unit vector of direction μ. The gauge transformation is represented by $\phi(n) \to \omega(n)\psi(n)$, $\phi^\dagger(n) \to \omega^\dagger(n)\psi^\dagger(n)$ and $U_\mu(n) \to \omega(n+\hat\mu)U_\mu(n)\omega^\dagger(n)$ and the covariant derivative is replaced by the finite difference $D_\mu\phi(n) = U^\dagger_\mu(n)\phi(n+\hat\mu) - \phi(n)$, in particular $D^2\phi(n) = \sum_\mu [U^\dagger_\mu(n)\phi(n+\hat\mu) + U_\mu(n-\hat\mu)\phi(n-\hat\mu) - 2\phi(n)]$. The partition function of the bare Euclidean theory is the path integral

$$Z = \int D[U]D[\phi^\dagger]D[\phi]e^{-S_L} \tag{14}$$

with the lattice action

$$S_L = \sum_n \sum_{\gamma'} a_{\gamma'} \mathrm{tr} U_{\gamma'}(n) + \sum_n \phi^\dagger(n) \sum_\gamma U^\dagger_\gamma(n)\phi(n+\gamma)b_\gamma + \sum_n V(\phi^\dagger(n)\phi(n)) \tag{15}$$

where γ' and γ denote closed and open paths respectively, up to length n_d, $n+\gamma$ stands for the lattice site where the path γ arrives upon starting at n and $U_\gamma(n)$ is the path ordered product of the link variables along this path. Time reversal invariance requires that for each path γ its time reversed version $\Theta\gamma$ be included in the sums with $a_{\Theta\gamma'} = a^*_{\gamma'}$, $b_{\Theta\gamma} = b^*_\gamma$, making the action S_L real.

Guided by the classical procedure [16] we construct n_d lattice field variables in the following manner [17]. We pick n^0 as the time coordinate on the lattice and regroup each n_d consecutive space-like hyper-surface into a single blocked time slices of the Euclidean space-time. The new fields, defined as functions of the new time variable are

$$\phi_j(t, \mathbf{n}) = \phi(n_d t + j, \mathbf{n}), \tag{16}$$

and

$$U_{j,\mu}(t, \mathbf{n}) = U_\mu(n_d t + j, \mathbf{n}) \tag{17}$$

where t is integer and $j = 1, \cdots, n_d$ and we write the action (15) as

$$S_L = \sum_t [L_s(t) + L_{km}(t) + L_{kg}(t)] \tag{18}$$

where

$$\begin{aligned} L_s(t) &= S_s[U(t), \phi^\dagger(t), \phi(t)], \\ L_{kg}(t) &= S_{kg}[U(t), U(t+1)], \\ L_{km}(t) &= \sum_{t,\mathbf{m},\mathbf{n}} \phi^\dagger_j(t+1, \mathbf{m})\Delta_{j,k}(\mathbf{m}, \mathbf{n}; U(t), U(t+1))\phi_k(t, \mathbf{n}) + c.c. \end{aligned} \tag{19}$$

Here $S_s[U, \phi^\dagger, \phi]$ includes field variables within a single blocked time slice, the expression $S_{kg}[U, U']$ collects the contributions to the gauge field action which contain the product of link variables U and U' belonging to two consecutive

blocked time slices. The matter inter-block kinetic term L_{km} represents the coupling of the matter field located at consecutive blocked time slices, containing the link variables U and U' of these two time slices and finally, c.c. stands for complex conjugate. One introduces the transfer matrix T defined by

$$\langle\phi^\dagger,\phi,U|T|\phi'^\dagger,\phi',U'\rangle = e^{L_{kg}[U,U']+\frac{1}{2}L_s[U,\phi^\dagger,\phi]+\frac{1}{2}L_s[U',\phi'^\dagger,\phi']+[\sum_{\mathbf{m},\mathbf{n}}\phi_j^\dagger(\mathbf{m})\Delta_{j,k}(\mathbf{m},\mathbf{n};U,U')\phi_k'(\mathbf{n})+c.c]} \tag{20}$$

which must be a positive operator in the physical subspace of the Fock-space. This condition will be expressed by the inequality

$$\langle 0|\mathcal{F}\Theta[\mathcal{F}]|0\rangle \geq 0 \tag{21}$$

holding for any local functional $\mathcal{F}$ including physical fields for positive t only. Note that this condition obviously excludes states with negative norm from the physical subspace.

The discussion of reversal parity in Section II makes clear that the norm of a state created by an operator on a time reversal invariant vacuum is determined by the time reversal parity of the operator in question. We shall use this relation to trace the physical states in the path integral of Eq. (14). The time reversal acts as

$$\Theta\mathcal{F}[\phi,\phi^\dagger,U] = \mathcal{F}[\Theta\phi,\Theta\phi^\dagger,\Theta U] \tag{22}$$

with

$$\begin{aligned}
\Theta\phi_j(n) &= \phi^\dagger_{\Theta j}(\Theta n),\\
\Theta\phi^\dagger_j(n) &= \phi_{\Theta j}(\Theta n),\\
\Theta U_{j,\mu}(n) &= \begin{cases} U^\dagger_{\Theta j,\mu}(\Theta n) & \mu = 1,2,3,\\ U_{\Theta j-1,\mu}(\Theta n) & \mu = 0, j < n_d,\\ U_{j,\mu}(\Theta n - \hat{0}) & \mu = 0, j = n_d,\end{cases}
\end{aligned} \tag{23}$$

where $\Theta j = n_d + 1 - j$ and the space-time coordinate $n = (t,\mathbf{n})$ transforms as $\Theta(t,\mathbf{n}) = (-t,\mathbf{n})$. We consider the site-inversion realization of time reversal, $t \to -t$, with odd n_d. Had we chosen even n_d we should have worked with the more involved link-inversion, $t \to 1-t$. We shall need functionals with well defined time inversion parity, $\Theta\mathcal{F}[\phi(n),\phi^\dagger(n),U(n)] = \tau_{\mathcal{F}}\mathcal{F}[\phi(\Theta n),\phi^\dagger(\Theta n),U(\Theta n)]$. The simplest way to obtain such functionals is to use the combinations

$$\begin{aligned}
\phi_{\tau,j}(t,\mathbf{n}) &= \frac{1}{2}[\phi_j(t,\mathbf{n}) + \tau\phi^\dagger_{\Theta j}(t,\mathbf{n})],\\
\phi^\dagger_{\tau,j}(t,\mathbf{n}) &= \frac{1}{2}[\phi^\dagger_j(t,\mathbf{n}) + \tau\phi_{\Theta j}(t,\mathbf{n})],\\
U_{\tau,j,\mu}(t,\mathbf{n}) &= \frac{1}{2}[U_{j,\mu}(t,\mathbf{n}) + \tau U^\dagger_{\Theta j,\mu}(t,\mathbf{n})], \quad \mu = 1,2,3
\end{aligned} \tag{24}$$

with $j = 1,\ldots,(n_d-1)/2$ of the local fields which display well defined time reversal parities. Note that the time reversal invariant combinations are the finite difference realization of the coordinates $\phi_{\tau,j}$ mentioned above. To achieve gauge invariance we perform the gauge transformation

$$\omega(n_d t + j,\mathbf{n}) = [U_0(n_d t + j - 1,\mathbf{n})\cdots U_0(n_d t + 1,\mathbf{n})]^\dagger \tag{25}$$

for $2 \leq j \leq n_d$ on the original lattice with simple time slices which cancels the time component of the gauge field within the block time slices and sets $U_{j,0}(t,\mathbf{n}) \to \mathbb{1}$ for $j < n_d$. The functional $\mathcal{F}$ with time reversal parity $\tau_{\mathcal{F}}$ may contain any products of fields as long as the product of their time reversal parity agree $\tau_{\mathcal{F}}$.

The proof of the inequality (21) proceeds in the usual manner [18], by the splitting of the action (18) into three pieces $S = L_s(0) + S_- + S_+$ with

$$\begin{aligned}
S_- &= \sum_{t<0}[L_s(t) + L_{km}(t) + L_{kg}(t)],\\
S_+ &= \sum_{t\geq 0}[L_{km}(t) + L_{kg}(t)] + \sum_{t>0} L_s(t)
\end{aligned} \tag{26}$$

We can now phrase an important condition of consistency, the time reversal invariance of the dynamics of our model. The time reversal invariance of the microscopic dynamics can be expressed by the equations $S_\pm[\Psi(t)] = \Theta[S_\mp[\Psi(t)]] =$

$S_{\mp}[\Psi(\Theta t)]$ and $S_0[\Psi] = S_0[\Psi(\Theta t)]$. Furthermore the vacuum of the theory may be nontrivial but is assumed to contain time reversal invariant condensates only.

We introduce the notation $\Psi = (U, \phi^\dagger, \phi)$ for the fields and write

$$\langle 0|\mathcal{F}\Theta\mathcal{F}|0\rangle = \int D[\Psi] e^{-S_0[\Psi]} e^{-S_+[\Psi]} \mathcal{F}[\Psi] e^{-S_-[\Psi]} \Theta[\mathcal{F}[\Psi]] \tag{27}$$

where the logarithm of the wave functional of the vacuum state is added to the actions $S_\pm[\Psi]$. To write this expression as an integral with positive definite integrands we use the time reversal invariance of three objects. (i) The time reversal acts in a nontrivial manner on the wave functionals of the states, it flips the sign of the variables with negative time reversal parity. The time reversal invariance of the vacuum state, $\Theta|0\rangle = |0\rangle$, allows us to include the vacuum state functional into the time reversed factor within the path integral,

$$\langle 0|\mathcal{F}\Theta\mathcal{F}|0\rangle = \int D[\Psi] e^{-S_0[\Psi]} e^{-S_+[\Psi]} \mathcal{F}[\Psi] \Theta \left[e^{-S_+[\Psi]} \mathcal{F}[\Psi] \right]. \tag{28}$$

(ii) The time reversal invariance of $S_0[\Psi]$ is used to rewrite our expectation value as

$$\langle 0|\mathcal{F}\Theta\mathcal{F}|0\rangle = \int D_{t=0}[\Psi] \int D_{t>0}[\Psi(t)] e^{-\frac{1}{2}S_0[\Psi(t)] - S_+[\Psi(t)]} \mathcal{F}[\Psi(t)] \Theta \mathcal{F}[\Psi(t)] \int D_{\Theta t>0}[\Psi(\Theta t)] e^{-\frac{1}{2}S_0[\Psi(\Theta t)] - S_+[\Psi(\Theta t)]} \tag{29}$$

(iii) Finally, we assume that the functional $\mathcal{F}[\Psi]$ has time-reversal parity $\tau_{\mathcal{F}}$ and find

$$\langle 0|\mathcal{F}\Theta\mathcal{F}|0\rangle = \tau_{\mathcal{F}} \int D_{t=0}[\Psi] \left(\int D_{t>0}[\Psi] e^{-\frac{1}{2}S_0[\Psi] - S_+[\Psi]} \mathcal{F}[\Psi] \right)^2 \tag{30}$$

which is positive for $\tau_{\mathcal{F}} = 1$.

The naively expected conditions for consistency, the time reversal invariance of the action and vacuum state are not enough for reflection positivity, we needed condition (ii) satisfied by each trajectory in the path integral. The simplest local way to ensure this condition together with (i) is to impose the boundary conditions

$$\Psi(t_f) = \tau_\Psi \Psi(t_i), \tag{31}$$

ie. using periodic or antiperiodic boundary conditions for time reversal even or odd variables, respectively. Such a generalized KMS construction restricts the path integration in a local, translationally invariant manner and eliminates the non-unitary runaway solutions. Full Poincaré invariance is recovered by imposing similar boundary conditions in each of space-time directions and performing the thermodynamical limit. Reflection positivity is assured within the Fock-space span by the local, time reversal invariant functionals of the fields acting on the time reversal invariant vacuum assuming the generalized KMS conditions.

Due to the compactness of the proof it is instructive to check the result in a simple particular case of a free real scalar theory defined by the Lagrangian $L = \phi D^{-1}(\Box)\phi/2$ and for the operators $\mathcal{F} = \partial_0^k \phi(x)$. The space of field configurations to integrate over is determined by the generalized KMS boundary conditions $\phi_j^{(n)}(t_f, \mathbf{x}) = (-1)^j \phi_j^{(n)}(t_i, \mathbf{x})$, imposed on the original field with $j = 0$ and the auxiliary fields for $j > 0$. The Lagrangian is quadratic in the auxiliary fields even in the presence of interactions and the equations of motion can be used to eliminate these variables. The result is a path integral for the original field satisfying the boundary conditions

$$\partial_0^j \phi(t_f, \mathbf{x}) = (-1)^j \partial_0^j \phi(t_i, \mathbf{x}). \tag{32}$$

Eq. (27) can now be written as

$$\partial_t^k \partial_{t'}^k \langle \phi(t, \mathbf{x}) \phi(t', \mathbf{x}) \rangle_{|t'=\Theta t} = \sum_n \partial_t^j \phi^{(n)}(t, \mathbf{x}) \partial_{t'}^j \phi^{(n)}(t', \mathbf{x})_{|t'=\Theta t} \lambda^{(n)} \tag{33}$$

where the eigenfunction $D(\Box)\phi^{(n)} = \lambda^{(n)}\phi^{(n)}$ satisfies the boundary condition (32). Since $\partial_t^k \phi^{(n)}(t, \mathbf{x}) = (-1)^k \partial_{t'}^k \phi^{(n)}(t', \mathbf{x})_{|t'=\Theta t}$ each contribution to the sum on the right hand side of Eq. (33) is positive for even k as long as the imaginary time path integral is convergent, $\lambda^{(n)} > 0$.

IV. SUMMARY

The elimination of some high energy particle modes poses interesting problems for the resulting low energy effective theory. An approximation which is necessary to render these theories useful is the truncation of the gradient expansion of the effective action. Two problems, related to the truncation of the effective theory have been considered in this work, the question of identifying the physical states of the effective theory and the proof that the low energy modes follow consistent dynamics.

The truncation of the gradient expansion of the effective dynamics generates new degrees of freedom which display a generalization of the KMS construction and are represented by periodic or antiperiodic trajectories in the path integral representation. It was shown in the framework of the Yang-Mills-Higgs model with higher order derivatives that reflection positivity holds for this theory within the Fock-space generated by the action of local expressions of time reversal invariant operators on the time reversal invariant vacuum when these boundary conditions are maintained. Note that such an extended boundary condition prescription can always be imposed on any quantum field theory of the type considered in this work because it modifies the propagators only when higher orders of the space-time derivatives occur in the action and the result holds for arbitrary value of the lattice spacing.

Further extensions of the argument presented above are needed to cover a bigger, more realistic family of effective theories. The higher order derivatives might appear in the terms containing higher order than quadratic of the matter fields. Furthermore, it is necessary to extend the argument for fermions whose exchange statistics and lattice formalism introduces further complexities.

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